Resource Theory of Entanglement

What is a property?

- Hermitian observable ⤳ what about time or phase?
- Something detectable via a POVM ⤳ what about more illusive properties like { "entanglement", "coherence", "Gaussianity", "Non-classicality" ....

Resource Theories – define a property by what it is not

- Define free operations
  - free states are those that can be generated by free operations
  - resource states are those that cannot be generated by free operations

They allow one to perform operations which were otherwise impossible

Can then ask questions about – ① Interconversion
② Quantification

Quantum Resource Theory

A quantum resource theory is defined by a set of free operations $\mathcal{F}$ which is closed under composition and contains the tracing out operation $\rho_{AB} \to Tr_B(\rho_{AB})$

The subset of free preparations $F_{states} \in \mathcal{F}$ are called 'Free states'

The set of states $R = F^c \cap D$ that are not free are called 'resource states'

$$D := \{\rho \in B(H) : \mathrm{tr}(\rho) = 1, \ \rho \geq 0\}$$

set of bounded linear operators that are +ve & normalised

skip?

A resource state allows us to enlarge the set of possible transformations

Simulation via resources

We can simulate $\mathcal{E} \notin F$ using a state σ if there is a free operation $\tilde{\mathcal{E}} \in F$ such that

$$\mathrm{Tr}\left(\tilde{\mathcal{E}}(\rho \otimes \sigma)\right) = \mathcal{E}(\rho) \quad \forall \rho$$

Resource theories induce a partial ordering determined by which states can be converted to which other states using free operations

$$\begin{tikzcd}
\rho_E \arrow[dr] & & \rho_D \arrow[dl] & & \\
& \rho_A \arrow[dr] & & & \\
& & \rho_B \arrow[dr, leftrightarrow] & & \rho_F \arrow[dl, leftrightarrow] \\
& & & \rho_C &
\end{tikzcd}$$

If $\rho_A \xrightarrow{\text{via free operations}} \rho_B$ we write $\rho_A \prec \rho_B$

Note if $\rho_A \prec \rho_B$ & $\rho_B \prec \rho_C \Rightarrow \rho_A \prec \rho_C$

(transitivity)

But the ordering is only partial as we do not require any two states can be related via $\succ$.

Resource Measure

A function $M : \mathcal{D} \to \mathbb{R}$ is called a resource if

$$M(\rho) \geq M(\mathcal{E}(\rho)) \quad \forall \quad \mathcal{E} \in F \ \& \ \rho$$

i.e. Free operations can only decrease the measure of the resource

Resource Theory of Entanglement

What cannot create entanglement?

- Local operations i.e. Alice applying quantum gates to her set up & Bob applying quantum gates to his set up

$$\boxed{LO : \quad \mathcal{E}(\ ) = \mathcal{E}_A \otimes \mathcal{E}_B}$$

- Classical communication i.e. between Alice & Bob
  - Classical information is information that is diagonal in the computational basis.
  - Communication involves a relabelling of who owns it

eg: $\rho_A \otimes |k\rangle\langle k|_A \otimes \sigma_B \rightarrow \rho_A \otimes |k\rangle\langle k|_B \otimes \sigma_B$

Or, more generally, it's the set of maps of the form

$$\boxed{CC : \quad C_{A \to B}(X) = \sum_n |n\rangle\langle n|_B \langle n|X|n\rangle_A}$$

Note that this allows Bob to do something locally that is conditional on the outcome of Alice's measurement

i.e. $\rho_{AB} \rightarrow \Sigma_i (M_i \otimes U_i) \rho_{AB} (M_i^\dagger \otimes U_i^\dagger)$

ex. show this can be done via a channel of the form

(hint controlled unitaries)

⇒ take these as the free operations for the "resource theory of entanglement"

Definition: LOCC - The resource theory of entanglement

The class LOCC consists of local operations & classical communications is generated by finite combinations of operations in LO & LC.
The resource theory with $F = LOCC$ is the resource theory of entanglement

The free states of the resource theory of entanglement
↳ i.e. those that can be generated by LOCC are states of the form

$$\rho_{AB} = \sum_k p_k \, \sigma_A^{(k)} \otimes \sigma_B^{(k)}$$

This is the set of seperable states D_{sep}

The resource states are then defined to be all non-seperable quantum states — i.e. the entangled states.

Interconvertibility of Resources

Could ask this question more generally for mixed states but it's much harder!

When is it possible to transform $|\psi\rangle_{AB} \xrightarrow{LOCC} |\phi\rangle_{AB}$ deterministically using only LOCC?

Example $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow |\theta\rangle_{AB} = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$

via LOCC?

Answer: Yes

Step 1: Alice performs a measurement

$$A_1 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{pmatrix} \qquad A_2 = \begin{pmatrix} \sin\theta & \\ & \cos\theta \end{pmatrix}$$

Kraus operators

$$M_A = \{ A_1^\dagger A_1 , A_2^\dagger A_2 \}$$

POVM elements

Step 2: If "2" Alice applies Bit flip

Step 3: Alice tells Bob the outcome & conditional on the measurement outcome Bob does nothing (if 1) or applies a bit flip (if 2).

$$|\phi^+\rangle \xrightarrow{M_A \otimes I} \begin{cases} \xrightarrow{A_1} \frac{1}{\sqrt{2}} |\cos(\theta)|00\rangle + \sin\theta|11\rangle| \\ \xrightarrow{A_2} \frac{1}{\sqrt{2}} (\sin(\theta)|00\rangle + \cos\theta|11\rangle) \\ \quad \downarrow X_A \otimes X_B \\ \quad \frac{1}{\sqrt{2}} (\sin(\theta)|11\rangle + \cos\theta|00\rangle) \end{cases} \rightarrow |\theta\rangle_{AB} \checkmark$$

But what about $|\phi^+\rangle \rightarrow |\phi\rangle_{AB}$

any 2 qubit state

(Actually additional complexity can be handled by local operations that generate phases.)

But what about $|\phi\rangle_{AB} \rightarrow |\phi^+\rangle_{AB}$? Harder! Not possible in general.

or $|\phi\rangle_{AB} \rightarrow |\psi\rangle_{AB}$?

⇒ LOCC defines a partial order on the set of pure bipartite quantum states.

i.e. given an initial entangled quantum state we can only reach certain other entangled states via LOCC.

Majorization Theory

An ordering relationship between two vectors $\underline{x}$ & $\underline{y}$

Let $\underline{x}^{\downarrow} = \underline{x}$ with elements reordered in decreasing order
$= (x_1^{\downarrow}, x_2^{\downarrow}, \dots, x_N^{\downarrow})$

$\underline{x}$ is majorized by $\underline{y}$ i.e. $\underline{x} \prec \underline{y}$ iff

1) $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{k}$ for all $k = 1, \dots, N$

2) $\sum_{j=1}^{N} x_j = \sum_{j=1}^{N} y_j$

e.g. $\underline{a} = (1/3, 1/3, 1/3)$

$\underline{b} = (1/6, 2/3, 1/6) \to \underline{b}^{\downarrow} = (2/3, 1/6, 1/6)$

$\underline{c} = (1, 0, 0)$

$\underline{d} = (1/2, 1/2, 1)$ → doesn't obey ②

$\underline{e} = (1/2, 1/2, 0)$

$1/3 \leq 2/3$, $2/3 \leq 5/6$, $1 = 1$ $\quad \underline{a} \prec \underline{b}$

$1/2 < 2/3$ & $1 > 5/6$ $\quad \underline{e}$ incomparible with $\underline{b}$ $\quad \underline{b} \prec \underline{c}$

$\underline{e} \prec \underline{c}$

$1/3 \leq 1/2$, $2/3 \leq 1 \to \underline{a} \prec \underline{e}$

check!

$$\left.\begin{tikzcd} & \underline{a} \arrow[dl] \arrow[dr] & \\ \underline{b} \arrow[dr] & & \underline{e} \arrow[dl] \\ & \underline{c} & \end{tikzcd}\right\} \text{partial order!}$$

More generally $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right) \prec \underline{p} \prec (1, 0 \dots)$

for all probability distributions $\underline{p}$.

Problem sheet will explore more properties.

Nielsen's Majorization Theorem

Let $|\phi\rangle_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

Define : $\lambda(\phi) := \{\text{eigenvalues of } \rho_A = \mathrm{Tr}_B(|\phi\rangle\langle\phi|_{AB})\}$

then $|\phi\rangle_{AB} \xrightarrow{LOCC} |\psi\rangle_{AB}$

iff $\lambda(\phi) \prec \lambda(\psi)$

(Proof in N&C)

In other words, the partial order induced by LOCC on the set of pure bipartite states coincides with the partial order from majorization theory for the marginal spectra $\lambda(\phi)$.

The maximally entangled state

$$|\Omega\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle_{AB}$$

is majorized by all other states & hence $|\Omega\rangle_{AB} \xrightarrow{LOCC} |\phi\rangle$ for any $|\phi\rangle$

as $\lambda(\Omega) = \left(\frac{1}{d}, \dots, \frac{1}{d}\right)$
& $\lambda(\Omega) \prec \lambda(\phi) \ \forall \phi$

Asymptotic Conversion rates

Given $N \to \infty$ copies of any state $|\phi\rangle_{AB}$ there exists a reversible interconversion

$$|\phi\rangle\langle\phi|^{\otimes N} \underset{LOCC}{\longleftrightarrow} |\psi^-\rangle\langle\psi^-|^{\otimes S(\rho_{\phi_A})N}$$

$\rho_{\phi_A} = Tr_B(|\phi\rangle\langle\phi|)$

proof $\quad |\phi\rangle^{\otimes N} \to |\psi^-\rangle^{\otimes K}$

Variants on this can be dealt with via local transformations

Let's assume $|\phi\rangle = \overbrace{\sqrt{1-\lambda}\,|00\rangle + \sqrt{\lambda}\,|11\rangle}^{\text{Schmidt decomp}}$

$$|\phi\rangle^{\otimes N} = (1-\lambda)^{N/2}\,|0\ldots0\rangle_A \otimes |0\ldots0\rangle_B$$
$$+ (1-\lambda)^{\frac{N-1}{2}}\lambda^{1/2}\left(|10\ldots0\rangle_A|10\ldots0\rangle_B + |01\ldots0\rangle_A + \ldots\right)$$
$$+ \ldots$$

$$|\phi\rangle^{\otimes N} = \sum_K \lambda^{K/2}(1-\lambda)^{\frac{N-K}{2}} \sum_\sigma |\sigma(\underbrace{11\ldots1}_{K}000)\rangle_A\,|\sigma(\underbrace{11\ldots1}_{K}000)\rangle$$

${}^N C_K$ terms in the sum spanning a K charge subspace.

Protocol for distilling $S(\rho_{\phi_A})N$ singlets with probability $p \to 1$ as $N \to \infty$

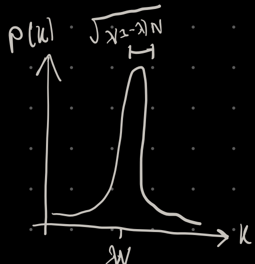
1) Local measurement at A

X_A counts the number of 1's in the charge

$$X_A\,|i_1 i_2 \ldots i_N\rangle = K\,|i_1 \ldots i_N\rangle$$

$$|\phi\rangle^{\otimes N} \xrightarrow{\chi_A} \frac{1}{{}^{n}C_k} \sum_{\pi} |\pi(\underbrace{1\,1\,..\,1}_{k}\,0\,..\,0)\rangle_A \otimes |\pi(\underbrace{1\,..\,1}_{k}\,0\,..\,0)\rangle_B$$

with probability $p_k = \underbrace{{}^{n}C_k\,(1-\lambda)^{N-k}\,\lambda^{k}}_{\text{Binomial distribution}}$



for large N the binomial distribution tends to the Gaussian distribution and is sharply peaked at

$$k = \lambda N \pm \sqrt{N(1-\lambda)\lambda}$$

Thus with high probability we will obtain the outcome

$$|k=\lambda N\rangle = \frac{1}{\sqrt{{}^{N}C_{\lambda N}}} \sum_{\pi} |\pi(\underbrace{1\,1\,1}_{\lambda N}\,..\,0\,0\,0)\rangle_A \otimes |\pi(1\,1\,1\,..\,0\,0\,0)\rangle_B$$

i.e. a uniform superposition over the λN subspace

2) Classical Communication

↳ Alice tells Bob she got the $x = \lambda N$ outcome

3) Local Unitaries

A & B now both know they share $|x = \lambda N\rangle$

Via the Stirling approx: $\log\left({}^{N}C_{\lambda N}\right) = N\left(-\lambda\log\lambda - (1-\lambda)\log(1-\lambda)\right)$

$$\Rightarrow \quad {}^{N}C_{\lambda N} = 2^{S(\phi_A)N}$$

i.e. the state $|x=\lambda N\rangle$ is a superposition of $\underbrace{2^{S(\phi_A)N}}$ orthogonal product vectors.

We can transform this state into $S(\phi_A)N$ maximally entangled qubits via the following local unitary applied to A & B independently.

$$
{}^N C_{\lambda N} \left\{ \begin{array}{lcl}
|11\ldots 10\ldots 0\rangle & \rightarrow & |000\ldots 0\rangle \otimes |0\ldots 0\rangle \\
|11\ldots 101\ldots 0\rangle & \rightarrow & |000\ldots 1\rangle \otimes |0\ldots 0\rangle \\
\vdots & & \quad\vdots \qquad\qquad \vdots \\
|100\ldots 1\ldots 1\rangle & \rightarrow & \underbrace{|111\ldots 1\rangle}_{\log_2({}^N C_{\lambda N}) \text{ qubits to label the } {}^N C_{\lambda N} \text{ states.}} \otimes \underbrace{|0\ldots 0\rangle}_{\text{Junk qubits}}
\end{array} \right.
$$

$\therefore$

$$|x=\lambda N\rangle \rightarrow \left(\frac{1}{\sqrt{{}^N C_{\lambda N}}} \sum_{i=1}^{{}^N C_{\lambda N}} |i\rangle \right) |\bar{0}\rangle \leftarrow \text{junk}$$

$$= |\phi^+\rangle^{\otimes S(\phi_A)N} |\bar{0}\rangle$$

What about the reverse protocol?

Can we dilute $S(\phi_A)N$ singlets into N copies of $|\phi\rangle_{AB}$?

Yes just run the protocol backwards!

Yes, via teleportation:

1) Local Operations: Alice creates N copies of $|\phi\rangle_{A_1A_2}$ locally

2) Compress the state on N qubits to $NS(\phi_A)$ qubits. Teleport these using $N S(\phi_A)$ singlet states $\Big\}$ 2CC

3) Local Operations: Bob uncompresses these to get $|\phi\rangle_{AB}^{\otimes N}$

Measure of Entanglement - E

Requirements for E

- An LOCC-monotone

  i.e. if $|\phi_1\rangle \xrightarrow{LOCC} |\phi_2\rangle \Rightarrow E(|\phi_1\rangle) \geq E(|\phi_2\rangle)$

- Take the singlet (max entangled qubit) as our gold standard

  i.e. $E(|\psi^-\rangle) = 1$ (Normalization)

- Extensivity : $E(|\phi\rangle^{\otimes k}) = k\, E(|\phi\rangle)$

There is a unique (normalized) measure of pure bipartite entanglement that satisfies monotonicity and extensivity - the von Neumann entropy.

Proof : $|\phi\rangle^{\otimes N} \overset{LOCC}{\longleftrightarrow} |\psi^-\rangle^{\otimes N S(\phi_A)}$

from monotonicity $E(|\phi\rangle^{\otimes N}) \leq E(|\psi^-\rangle^{\otimes N S(\phi_A)})$

$E(|\psi^-\rangle^{\otimes N S(\phi_A)}) \geq E(|\phi\rangle^{\otimes N})$

$\Rightarrow E(|\phi\rangle^{\otimes N}) = E(|\psi^-\rangle^{\otimes N S(\phi_A)})$

Extensivity $\Rightarrow N\, E(|\phi\rangle) = N S(\phi_A)\, E(\underbrace{|\psi^-\rangle}_{1})$ "Gold Standard"

$\Rightarrow \boxed{E(|\phi\rangle) = S(\phi_A)}$